Difference representation of one binomial expansion.

$(\sqrt{2}\ -\ 1)^n = \sqrt{m}\ - \sqrt{m-1}$

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Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$.

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We intend not only to prove that $(\sqrt{2}-1)^n$ for any $n \in \mathbb{N}$ can be represented in the form $\sqrt{m_n}-\sqrt{m_n-1}$, where $m_n \in \mathbb{N}$ but also to define m_n recursively.

For any $n \in \mathbb{N}$ let a_n and b_n be positive and negative parts of binomial expansion of $(\sqrt{2}\ -\ 1)^n$.We will prove that $a_n^2, b_n^2 \in \mathbb{N}$ and $a_n^2 - b_n^2 = 1$ for any $n \in \mathbb{N}$ and we thereby prove that $(\sqrt{2}-\ 1)^n = \sqrt{m_n}-\sqrt{m_n-1}$, where $m_n = a_n^2 \in \mathbb{N}$.

Since $(\sqrt{2}\ -\ 1)^{n+1} = (\sqrt{2}\ -\ 1)(\sqrt{2}\ -\ 1)^n = (\sqrt{2}\ -\ 1)(a_n - b_n) =$ $\sqrt{2}a_n + b_n - \left(a_n + \sqrt{2}b_n\right)$ we obtain $\begin{cases} a_{n+1} = \sqrt{2}a_n + b_n \\ b_{n+1} = a_n + \sqrt{2}b_n \end{cases}, n \in \mathbb{N}$,

where $a_1 = \sqrt{2}, b_1 = 1$.

Also, we can set $a_0 := 1, b_0 := 0$ since $(\sqrt{2}\ -\ 1)^0 = 1 = 1 - 0$

Since $b_n = a_{n+1} - \sqrt{2}a_n$ for any $n \in \mathbb{N}$ then $b_{n+1} = a_n + \sqrt{2}b_n$ becomes

$a_{n+2} - \sqrt{2}a_{n+1} = a_n + \sqrt{2}\left(a_{n+1} - \sqrt{2}a_n\right) \iff$

$a_{n+2} - 2\sqrt{2}a_{n+1} + a_n = 0, n \in \mathbb{N}\cup\{0\} \iff$

(1) $\quad a_{n+1} - 2\sqrt{2}a_n + a_{n-1} = 0, n \in \mathbb{N}$.

Thus, a_n is defined in $\mathbb{N}\cup\{0\}$ by recurrence **(1)** with initial conditions $a_0 = 1, a_1 = \sqrt{2}$.

Since $a_{n+1}^2 - a_{n+2}a_n = a_{n+1}\left(2\sqrt{2}a_n - a_{n-1}\right) - \left(2\sqrt{2}a_{n+1} - a_n\right)a_n =$

$a_n^2 - a_{n+1}a_{n-1}, n \in \mathbb{N}$ and $a_2 = 3$ then

$a_n^2 - a_{n+1}a_{n-1} = a_1^2 - a_2a_0 = 2 - 3 = -1, n \in \mathbb{N}$.

Hence, $a_{n+1}a_{n-1} = a_n^2 + 1, n \in \mathbb{N}$ and noting that

$(a_{n+1} + a_{n-1})^2 = \left(2\sqrt{2}a_n\right)^2 \iff a_{n+1}^2 + a_{n-1}^2 + 2a_{n+1}a_{n-1} = 8a_n^2$

we obtain

$a_{n+1}^2 + a_{n-1}^2 + 2a_n^2 + 2 = 8a_n^2 \iff a_{n+1}^2 - 6a_n^2 + a_{n-1}^2 = -2, n \in \mathbb{N}$

and since $a_0^2 = 1, a_1^2 = 2$ this immediately implies that $a_n^2 \in \mathbb{N}$.

By the similar way, since $b_{n+1} - 2\sqrt{2}b_n + b_{n-1} = 0, n \in \mathbb{N}$ and $b_0 = 0, b_1 = 1$

we obtain $b_n^2 - b_{n+1}b_{n-1} = b_1^2 - b_2b_0 = 1 \iff b_{n+1}b_{n-1} = b_n^2 - 1$ and, therefore, $b_{n+1}^2 - 6b_n^2 + b_{n-1}^2 = 2, n \in \mathbb{N}$ and this immediately implies that $b_n^2 \in \mathbb{N}$. Since $a_0^2 - b_0^2 = 1, a_1^2 - b_1^2 = 1$ and for any $n \in \mathbb{N}$, assuming $a_{n-1}^2 - b_{n-1}^2 = a_n^2 - b_n^2 = 1$,we obtain $a_{n+1}^2 - b_{n+1}^2 = 6\left(a_n^2 - b_n^2\right) - \left(a_{n-1}^2 - b_{n-1}^2\right) - 4 = 6 - 1 - 4 = 1$ then, by Math Induction, $a_n^2 - b_n^2 = 1$ for any $n \in \mathbb{N}\cup\{0\}$.

Thus, $(\sqrt{2}-\ 1)^n = \sqrt{m_n} - \sqrt{m_n - 1}$,where $m_{n+1} = 6m_n - m_{n-1} - 2, n \in \mathbb{N}$ and $m_0 = 1, m_1 = 2$.